The Algebraic Interpretation of Classical and Intuitionistic Quantifiers

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Please Note: The presentation had many verbal comments and explanations that are not recorded in the slides.

What is a Partial Ordering?

$$x \leq x$$

$$x \leq y \,\&\, y \leq z \Rightarrow x \leq z$$

$$x \leq y \,\&\, y \leq x \Rightarrow x = y$$

What is a Lattice?

$$0 \leq x \leq 1$$

$$x \vee y \leq z \Leftrightarrow x \leq z \,\&\, y \leq z$$

$$z \leq x \wedge y \Leftrightarrow z \leq x \,\&\, z \leq y$$

What is a Complete Lattice?

$$\bigvee_{i\in I} x_i \leq y \Leftrightarrow (\forall i\in I)\ x_i \leq y$$

$$y \leq \bigwedge_{i\in I} x_i \Leftrightarrow (\forall i\in I)\ y \leq x_i$$

Note:

$$\bigwedge_{i\in I} x_i = \bigvee\{y \mid (\forall i\in I)\ y \leq x_i\}$$

What is a Heyting Algebra?

$$x\wedge y\leq z \Leftrightarrow x\leq y\rightarrow z$$

What is a Boolean Algebra?

$$x\wedge y\leq z \Leftrightarrow x\leq y\rightarrow z$$

$\neg\neg x = x$, where $\neg x = x\rightarrow 0$

Every Heyting Algebra is Distributive

$$x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z)$$

Every Complete Heyting Algebra is Completely Distributive

$$x\wedge\bigvee_{i\in I} y_i = \bigvee_{i\in I}(x\wedge y_i)$$

Some Abbreviations

Ha = Heyting Algebra

cHa = Complete Heyting Algebra

Ba = Boolean Algebra

cBa = Complete Boolean Algebra

An Important Theorem

Theorem. Every complete lattice which is $(\wedge\bigvee)$-distributive is a cHa.

Hint: $y \to z = \bigvee\{x \mid x \wedge y \leq z\}$

Corollary. A non-empty $(\wedge\bigvee)$-closed subset of a cHa is also a cHa.

Rasiowa-Sikorski's Motto

Die Mathematiker sind eine Art Franzosen: redet man zu ihnen, so übersetzen sie es in ihre Sprache und dann ist es alsobald ganz etwas Anderes.

—J. W. von Goethe

cHa Semantics for Logic

$$\langle\!\langle aRb \rangle\!\rangle = \text{given}$$

$$\langle\!\langle \Phi \wedge \Psi \rangle\!\rangle = \langle\!\langle \Phi \rangle\!\rangle \wedge \langle\!\langle \Psi \rangle\!\rangle$$

$$\langle\!\langle \Phi \vee \Psi \rangle\!\rangle = \langle\!\langle \Phi \rangle\!\rangle \vee \langle\!\langle \Psi \rangle\!\rangle$$

$$\langle\!\langle \Phi \to \Psi \rangle\!\rangle = \langle\!\langle \Phi \rangle\!\rangle \to \langle\!\langle \Psi \rangle\!\rangle$$

$$\langle\!\langle \exists x.\Phi(x) \rangle\!\rangle = \bigvee_{a \in A} \langle\!\langle \Phi(a) \rangle\!\rangle$$

$$\langle\!\langle \forall x.\Phi(x) \rangle\!\rangle = \bigwedge_{a \in A} \langle\!\langle \Phi(a) \rangle\!\rangle$$

Semantical Completeness

A sentence Φ is provable in intuitionistic first-order logic if, and only if, $\langle\!\langle \Phi \rangle\!\rangle = 1$ whatever the interpretation in a cHa.

The proof from left to right is obvious!

Generic Completeness

There is (relative to the choice of language) a ***single*** cHa (cBa) such that if $\langle\!\langle \Phi \rangle\!\rangle = 1$ for this algebra, then Φ is provable in intuitionistic (classical) first-order logic.

The proof goes through the Lindenbaum algebra and the MacNeille completion of lattices.

MacNeille Completion

There are many (equational) varieties between Ha's and Ba's.

However, the completeness process only puts us in the ***same*** variety in the two extreme cases.

But, it ***does work*** for the extension to ***modal*** S4 Ha's and Ba's.

Modal S4 Algebras

$$\Box 1 = 1$$

$$\Box\Box x = \Box x \leq x$$

$$\Box(x \wedge y) = \Box x \wedge \Box y$$

The second two can be combined:

$$\Box x = \bigvee\{y \mid y = \Box y \leq x\}$$

Boole *vs.* Heyting

Theorem. For every cBa $\mathbf{B}$, there is a cHa $\mathbf{H}$ such that
$$\mathbf{B} = \{\neg\neg x \mid x \in \mathbf{H}\}.$$

Theorem. For every cHa $\mathbf{H}$, there is an S4 cBa $\mathbf{B}^{\Box}$ such that
$$\mathbf{H} = \{\Box x \mid x \in \mathbf{B}^{\Box}\}.$$

Relation Algebras

For every cBa **B**, the power $\mathbf{B}^{\mathbb{N}\times\mathbb{N}}$ becomes a ***relation algebra*** via:

$$(R;S)(i,j) = \bigvee_{k\in\mathbb{N}}(R(i,k)\wedge S(k,j))$$

$$R\breve{}(i,j) = R(j,i)$$

$$1'(i,j) = \bigvee\{1 \mid i = j\}$$

So, what about cHa's?

My Unanswerable Question

Why did not the Warsaw/Berkeley school consider Boolean-valued models of ***higher-order logic*** (and set theory) already in 1954/55?

Another Title for This Lecture

Ten Years Too Late!

The End